First Paper HW will be posted later today.

Section 1.4 Separable Equations

consider $\frac{dy}{dx} = f(x,y)$ if $f(x,y)$ does not contain variable y then solve using antiderivatives

ex. given $\frac{dy}{dx} = -6x$

then $y(x) = \underbrace{\int \frac{dy}{dx}\, dx}$ $= -6\int x\, dx$

$= -3x^2 + C$

ex. given $\frac{dy}{dx} = -6xy$ <u>cannot</u> be solved by just integrating both sides

new method: <u>separation of variables</u>

$\frac{1}{y}\, dy = -6x\, dx$ find $y(x)$

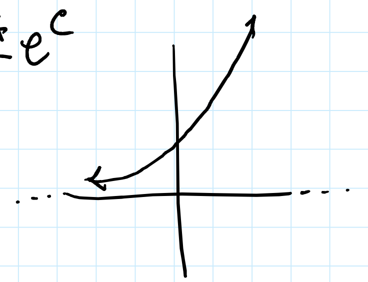
$\int \frac{1}{y}\, dy = -6\int x\, dx$

$\ln|y| = -3x^2 + C$

$|y| = e^{-3x^2+C}$

$y = \pm e^{-3x^2} e^{C}$ let $A = \pm e^{C}$

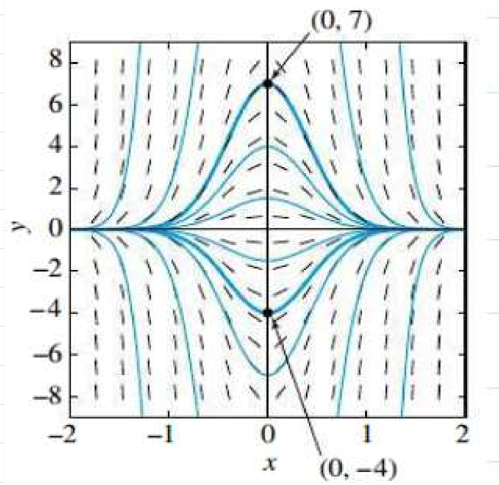
$y = Ae^{-3x^2}$ because of HA UND @ $A = 0$



when $A = 1$ $y = e^{-x^2}$

add initial condition $y(0) = 7$ $7 = Ae^0$, $7 = A$ $\Rightarrow y = 7e^{-3x^2}$

or given $y(0) = -4 \Rightarrow 4 = A \Rightarrow y = -4e^{-3x^2}$



$A \neq 0$ by def'n however when $A = 0 \Rightarrow y(x) \equiv 0$

is defined as

because $y = 0 \cdot e^{-3x^2} \Rightarrow 0$

conclude: because of how A is defined : $A = \pm e^c$

$y(x) = Ae^{-3x^2}$ provides sol'n for $\frac{dy}{dx} = -6xy$

for <u>ALL</u> values of A including $A=0$

important: didn't initially capture <u>all</u> solutions since divided by y in solving process

inadvertantly made assumption $y \neq 0$

1ST order DE is considered separable when $f(x,y)$ CBW as <u>product</u> of function of x and function of y

$$\frac{dy}{dx} = f(x,y) = g(x) \cdot k(y) = \frac{g(x)}{h(y)}$$

define $k(y) = \frac{1}{h(y)}$

then $\frac{dy}{dx} = \frac{g(x)}{h(y)} \longrightarrow h(y)\,dy = g(x)\,dx$

separable

also CBW as

$$h(y)\frac{dy}{dx} = g(x)$$

integrate WRT x

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

$\downarrow$ integrate

$$\int h(y)\frac{dy}{\cancel{dx}}\,\cancel{dx} = \int g(x)\,dx$$

$$\int h(y)\,dy = \int g(x)\,dx$$

$\downarrow$ antiderivative

$$H(y) = G(x) + C$$

$\uparrow$ technically $y(x)$

ex. solve $\dfrac{dy}{dx} = \dfrac{4-2x}{3y^2-5}$

rational fraction

$$= 4-2x \cdot \frac{1}{\boxed{3y^2-5}}$$

$$= g(x) \cdot k(y) \leftarrow \text{product}$$

$3y^2 - 5 = 0$

$3y^2 = 5$

$y^2 = \frac{5}{3}$

$y = \pm\sqrt{\frac{5}{3}} \leftarrow$ HA

conclude: DE will <u>not</u> have a solution @ $y = \pm\sqrt{\frac{5}{3}}$

i.e. no solution curves of DE will cross horizontal lines $y = \sqrt{\frac{5}{3}}$ or $y = -\sqrt{\frac{5}{3}}$

bc $\frac{dy}{dx}$ is UND along those lines

now we can find the general soln of DE:

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$

$$\int (3y^2-5)\,dy = \int (4-2x)\,dx$$

$$y^3 - 5y = 4x - x^2 + C$$

arb. constant

$\leftarrow$ can't readily solve for y, instead solve for C

$$C = y^3 - 5y - (4x - x^2)$$

$\downarrow$

$F(x,y)$

sol'n curves contained here (but won't cross $y = \pm\sqrt{\frac{5}{3}}$

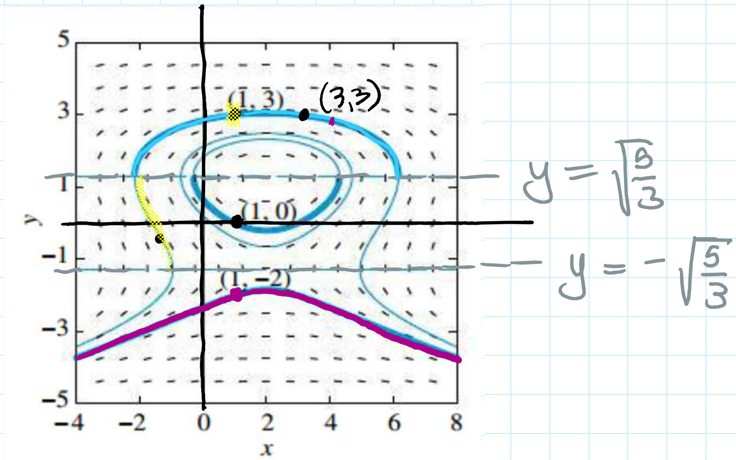
• given $y(1) = 3 \rightarrow 27 - 15 - (4-1)$

sol'n curves contained here (but won't cross $y = \pm\sqrt{\frac{5}{3}}$

- given $y(1) = 3 \Rightarrow 27 - 15 - \left(\frac{4}{3} - 1\right)$
  $= 12 - \left(\frac{4}{3}-1\right) \Rightarrow 9 = C$

  $\therefore$ specific sol'n curve lies on $y^3 - 5y - (4x - x^2) = 9$

  to correct for $y = \sqrt{\frac{5}{3}}$, restrict sol'n curve to $y > \sqrt{\frac{5}{3}}$



- given $y(1) = -2$ $\quad C = -5 \quad$ restrict sol'n curve to $y < -\sqrt{\frac{5}{3}}$

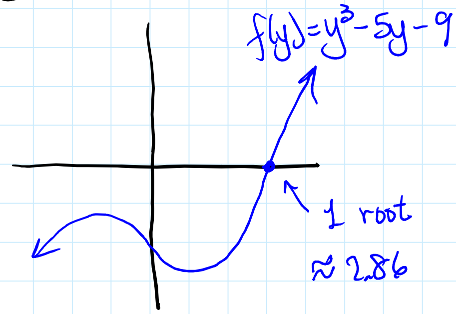
- given $y(1) = 0$ $\quad$ restrict sol'n curve to $-\sqrt{\frac{5}{3}} < y < \sqrt{\frac{5}{3}}$

- however, when $x = 4$: $\quad y^3 - 5y - \cancel{(4(4) - 4(4))}^{0} = 9 \quad \leftarrow$ only y

  $\therefore f(y) = \underset{\uparrow \text{cubic}}{y^3} - 5y - 9 = 0$

  $y(4) \approx 2.86$

  also $y(0) \approx 2.86$



- given $y(3) = 3 \Longrightarrow 9 = C$ $\quad$ on same curve as $y(1) = 3$

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given sol'n $y = y(x)$ of DE then $K(x,y) = 0$ is called an *implicit* sol'n of DE

note: $K(x,y) = 0$ may not satisfy an initial condition

ex. given $x^2 + y^2 = 4$ $\quad$ circle $r = 2$ center @ $(0,0)$

ex. given $x^2+y^2=4$ circle $r=2$ center @ $(0,0)$

$$\frac{d}{dx}\left(x^2+y^2=4\right)$$

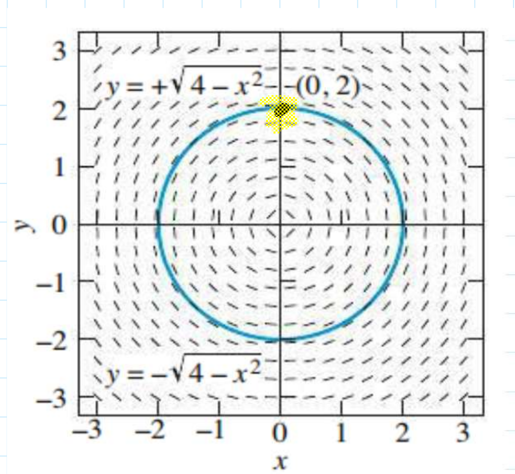
$$2x+2y\frac{dy}{dx}=0 \longrightarrow x+y\frac{dy}{dx}=0$$

or

$$x+yy'=0 \longleftarrow \text{DE}$$

Conclude: $x^2+y^2=4$ is an implicit sol'n to $x+yy'=0$

$$y_1=+\sqrt{4-x^2} \qquad y_2=-\sqrt{4-x^2}$$



• given initial condition $y(0)=2$ only $y_1=\sqrt{4-x^2}$ work

ex. find all sol'ns $\frac{dy}{dx}=6x(y-1)^{2/3}$

$$\int\frac{1}{(y-1)^{2/3}}\,dy=6\int x\,dx$$

$$\int(y-1)^{-2/3}\,dy=3x^2+C_1 \quad \text{where } C_1 \text{ is arb. constant}$$

$$\frac{3}{3}(y-1)^{1/3}=\frac{3x^2}{3}+\frac{C_1}{3} \quad \text{solve for } y$$

$$(y-1)^{1/3}=x^2+C_2 \quad \text{where } C_2=\frac{C_1}{3}$$

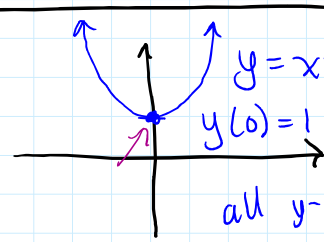
$$y-1=(x^2+C)^3$$

$$\boxed{y=(x^2+C)^3+1}$$

continuation 4 feb:

• when $C=0$: $y=(x^2)^3+1$

$$y=x^6+1$$



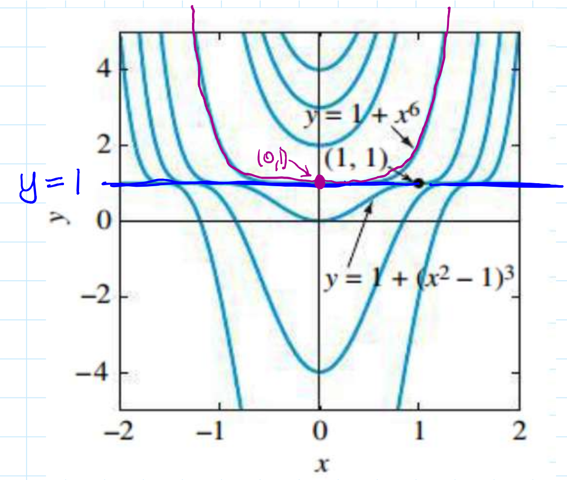
$y=x^6+1$

$y(0)=1$ is on curve

all y-values are positive

$c=0$

- when $C > 0$: lies above $y = 1$
- when $C < 0$: can dip below $y = 1$
- no value of C that return $y \equiv 1$
  (sol'n got "lost" in separation)



it's what's a singular solution —
cannot be obtained from general sol'n through an arbitrary choice of C

note: $y(x) \equiv 1$ (horizontal line) satisfies $y(1) = 1$

AND

$y(x) = (x^2-1)^3 + 1$ also satisfies $y(1) = 1$

only if we consider it outside of separable eq'n